Upper bound for product of two medians.

Problem 5291,**SSMJ February 2014**, **Proposed by Arkady Alt.**

Let m_a, m_b be medians of a triangle with sidelengths a, b, c. Prove that

$$m_a m_b \le \frac{2c^2 + ab}{4}.$$

Solution 1.

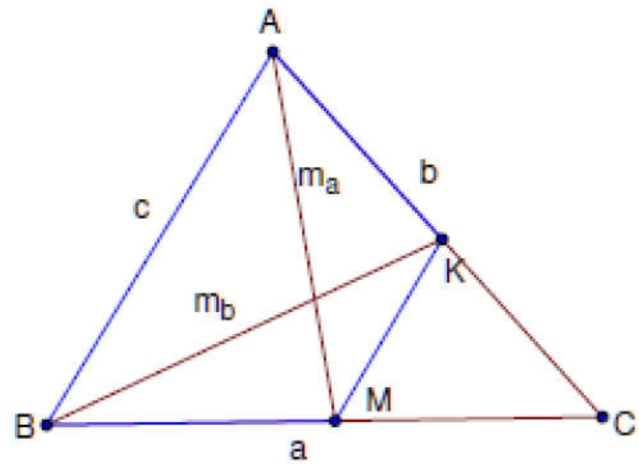
Since $m_a^2 = \dfrac{2(b^2+c^2)-a^2}{4}, m_b^2 = \dfrac{2(c^2+a^2)-b^2}{4}$ then

$16\left(\left(\frac{2c^2+ab}{4}\right)^2 - m_a^2 m_b^2\right) = (2(b^2+c^2)-a^2)(2(c^2+a^2)-b^2) -$

$(2c^2+ab)^2 = 2\left((a^2-b^2)^2 - c^2(a-b)^2\right) =$

$2(a-b)^2(a+b+c)(a+b-c) \ge 0.$

Solution 2.



Applying Ptolemy's Inequality to quadrilateral (trapezoid) $AKMB$ we obtain

$BK \cdot AM \le AB \cdot MK + AK \cdot BM \Leftrightarrow m_a m_b \le c \cdot \frac{c}{2} + \frac{b}{2} \cdot \frac{a}{2} \Leftrightarrow m_a m_b \le \frac{2c^2+ab}{4}.$